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On the Uniqueness of the Best Uniform Extended Totally Positive Monospline

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INTRODUCTION

In this paper we consider monosplines whose kernel is extended totally positive. We complement our results [1] on the existence of a monospline of minimal uniform norm by showing it is unique. As an application of these results we discuss the kernels $K_\alpha(x, \xi) = (1 - x\xi)^{-\alpha}$ and $K(x, \xi) = \exp(x\xi)$.

MAIN RESULTS

Let $K(x, \xi)$ be a real valued kernel. For a given set of odd integers $m = \{m_1, ..., m_t\}$, an integer $n \geqslant 1$ and an interval $[a, b] \subset (0, 1)$ we consider the set of all monosplines of the form:

$$M(A, x) = \int_a^b K(x, \xi)\, d\xi + \sum_{j=0}^{n-1} a_j K^j(x, 0) + \sum_{i=1}^{t} \sum_{j=0}^{m_i-1} a_{ij} K^j(x, \xi_i) \qquad (1)$$

where

$$0 \leqslant \xi_1 < \xi_2 < \cdots < \xi_t \leqslant d',\ a < b < d',$$

and

$$K^j(x, \xi_i) = \frac{\partial^j}{\partial \xi^j} K(x, \xi)|_{\xi=\xi_i}.$$

Further, with $N = n + \sum_{i=1}^{t} (m_i + 1)$, we set

$$A = (a_0, ..., a_{n-1}, a_{1,0}, ..., a_{t,m_t-1}, \xi_1, ..., \xi_t) \subset R^N.$$

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Setting $\|f\| = \max_{x\in[0,1]} |f(x)|$, our first result is:

THEOREM 1. *There is a monospline of the form* (1) *having minimal uniform norm on* [0, 1]. *Further, any such monospline* $M(A, x)$ *has the following property*:

$M(A, x)$ *alternates* N *times, i.e., there are* $N+1$ *points* $0 = x_0 < x_1 < \cdots < x_N = 1$ such that

$$M(A, x_i) = (-1)^{i+N} \| M(A, x)\| \qquad i = 0, 1,..., N.$$

Proof. Using [1, Theorems 1.1, 1.2] the only thing remaining to be proved is that $x_0 = 0$, $x_N = 1$. This however, follows from the first sentence of Lemma 2 below. ▮

In this paper we will be concerned with the uniqueness of the monospline of minimal uniform norm described by Theorem 1. For this purpose we will consider the following problems:

A. Given numbers $\{l_i\}_{i=0}^N$ where $\text{sgn}(l_k - l_{k-1}) = (-1)^{N+k}$, $k = 1,..., N$, find $A \in R^N$, $E \in R^+$ and $0 = x_0 < x_1 < \cdots < x_N = 1$ such that;

$$\begin{aligned} M(A, x_i) &= El_i, \qquad & i &= 0, 1,..., N, \\ M'(A, x_i) &= 0, \qquad & i &= 1,..., N-1. \end{aligned} \tag{2}$$

B. Show that the solution to problem A is unique.

[For the remainder of this paper we will consider the numbers $\{l_i\}_{i=0}^N$, subject to the above conditions, as fixed.].

To treat this problem we need a number of preliminary results.

DEFINITION 1. Let $K_x^j(x, \xi_k) = (\partial^{j+1}/\partial x \partial \xi^j) \quad K(x, \xi)|_{\xi=\xi_k}$ and $\bar{\xi} = (\xi_1, ..., \xi_t)$ we then define the symbols

$$K(x, \bar{\xi})_{N-1} = [K^1(x, 0), K^2(x, 0),..., K^{n-1}(x, 0), K(x, \xi_1),..., K^{m_1-1}(x, \xi_1), \\ K(x, \xi_2),..., K^{m_t-1}(x, \xi_t), K^{m_1}(x, \xi_1),..., K^{m_t}(x, \xi_t)]$$

$$K_x(x, \bar{\xi})_{N-1} = [K_x^1(x, 0), K_x^2(x, 0),..., K_x^{n-1}(x, 0), K_x(x, \xi_1),..., K_x^{m_1-1}(x, \xi_1), \\ K_x(x, \xi_2),..., K_x^{m_t-1}(x, \xi_t), K_x^{m_1}(x, \xi_1),..., K^{m_t}(x, \xi_t)]$$

(Note the term $K(x, 0)$ is missing from $K(x, \bar{\xi})_{N-1}$, and $K_x(x, 0)$ is missing from $K_x(x, \bar{\xi})_{N-1}$.)

LEMMA 1. *The components of the* $N-1$ *row vector* $K_x(x, \bar{\xi})_{N-1}$ *form an*

We make the following assumptions for the remainder of the paper.

I. $K(x, \xi)$ is an Extended Totally Positive kernel of order N in both x and ξ in $(c, d) \times [0, d']$ (see [8, p. 6]), where $c < 0 < 1 < d'$.

II. $K(x, 0)$ is a constant.

Extended Complete Tchebycheff System in $(c, d) \times [0, d']$ *of order* $N - 1$ (*see* [8, *p.* 375]). *We will abbreviate this by saying that* $K_x(x, \xi)$ *is an ECT system.*

Proof. Since $K(x, 0)$ is a constant, the result follows by using the concept of the reduced system (cf. [8, pp. 376-7 especially eq. (1.3), (1.4) and also Theorem 1.1]). ∎

LEMMA 2. *$M'(A, x)$ has at most $N - 1$ zeros in* [0, 1] *counting multiplicities. Further if $M'(A, x)$ has $N - 1$ zeros in* [0, 1] *counting multiplicities then its free knots satisfy*:

$$a < \xi_1 < \xi_2 < \cdots < \xi_1 < b$$

and $a_{i,mi-1} < 0$ $i = 1,..., t$; $M'(A, 1) > 0$.

Proof. Since $K_x(x, \xi)$ is an *ECT* system on $(c, d) \times [0, d']$ it follows by [1, Lemma 1.3] that $M'(A, x)$ has at most $N - 1$ sign changes in (c, d), i.e., there exist at most N points $c < x_1 < \cdots < x_N < d$ such that $M'(A, x_i)/M'(A, x_{i+1}) < 0$ $i = 1,..., N - 1$. From this it follows by a perturbation argument that $M'(A, x)$ has at most $N - 1$ zeros counting multiplicities in [0, 1] (cf. [2, Lemma 3, Theorem 2].)

LEMMA 3. *If $M'(A, x)$ has $N - 1$ zeros in* [0, 1] *counting multiplicities, then the components of A* (*with a_0 assumed bounded*) *are bounded.*

Proof. We argue by contradiction. Assume there exists a sequence

$$M'_\nu(x) = \int_a^b K_x(x, \xi)\, d + \sum_{j=1}^{n-1} a_j^\nu K_x^j(x, 0) + \sum_{i=1}^{t} \sum_{j=0}^{m_i-1} a_{ij}^\nu K_x^j(x, \xi_i^\nu)$$

with A^ν unbounded. By Lemma 2 the ξ_i^ν are bounded. Let $a^\nu = \max(\max_{j \neq 0} | a_j^\nu |, \max_{i,j} | a_{ij}^\nu |)$; therefore, $a^\nu \to \infty$.

By picking a subsequence, which we do not relabel we find

$$\frac{M'_\nu(x)}{a^\nu} \to s(x) = \sum_{j=1}^{n-1} b_j K_x^j(x, 0) + \sum_{i=1}^{t} \sum_{j=0}^{m_i-1} b_{ij} K_x^j(x, \xi_i); \; s(x) \not\equiv 0$$

where the convergence is in $C^N[0, 1]$, i.e. the functions together with their first N derivatives uniformly converge to $s(x)$ and its first N derivatives.

Since $M'_\nu(x)$ has $N - 1$ zeros on [0, 1] it then follows by Rolle's theorem that $s(x)$ has $N-1$ zeros on $[0, 1]$. However since $K_x(x, \xi)$ is an *ECT* system, $s(x)$ can have at most $N-t-2$ zeros. This contradiction proves the result. ∎

DEFINITION 2. Let $\Lambda = \{A \in R^N \mid M'(A, x)$ has a set of $N-1$ distinct zeros $0 < x_1(A) < x_2(A) < \cdots < x_{N-1}(A) < 1\}$.

The approach we use to solve (2) is to show that a solution to a particular system of differential equations is also a solution of (2). Our methods are in the spirit of the work of Fitzgerald and Schumaker [5].

We proceed in the following way. Start with any $A \in \Lambda$ (by Theorem 1 we know Λ is not empty). Let

$$M(A, x_k(A)) = d_k \qquad k = 0,\ldots, N \qquad \begin{matrix} x_0(A) = 0 \\ x_N(A) = 1 \end{matrix}$$

where

$$M'(A, x_k(A)) = 0 \qquad k = 1,\ldots, N-1. \tag{2'}$$

By Lemma 2 $\operatorname{sgn}(d_k - d_{k-1}) = (-1)^{N+k}$ $k = 1,\ldots, N$.

We now seek the solution of the system of $N+1$ differential equations

$$\frac{d}{ds} M(A(s), x_k(A(s))) = \frac{d}{ds} E(s)\, l_k - d_k \qquad k = 0,\ldots, N \tag{3}$$

with $x_k(A(s))$ determined from

$$\begin{gathered} M'(A(s), x_k(A(s))) = 0 \qquad k = 1,\ldots, N-1 \\ x_0(A(s)) = 0,\ x_N(A(s)) = 1. \end{gathered} \tag{3'}$$

Using (3′), (3) becomes

$$\begin{aligned} &\sum_{j=0}^{n-1} \frac{da_j(s)}{ds} K^j(x_k(A(s)), 0) + \sum_{i=1}^{t} \sum_{j=0}^{m_i-1} \frac{da_{ij}(s)}{ds} K^j(x_k(A(s)), \xi_i(s)) \\ &\qquad + \sum_{i=0}^{t} \sum_{j=1}^{m_i-1} a_{ij}(s)\, K^{j+1}(x_k(A(s)), \xi_i(s)) \frac{d\xi_i}{ds} - \frac{d}{ds} E(s)\, l_k \\ &= -d_k \qquad k = 0,\ldots, N. \end{aligned}$$

The initial conditions are $A(0) = A$, $E(0) = 0$.

If (3) and (3′) can be solved, then for arbitrary s the solution will have the form:

$$\begin{gathered} M(A(s), x_k(A(s))) = E(s)\, l_k + (1-s)\, d_k \qquad k = 0,\ldots, N \\ M'(A(s), x_k(A(s))) = 0,\ k = 1,\ldots, N-1;\ x_0(A(s)) = 0,\ x_N(A(s)) = 1. \end{gathered} \tag{5}$$

So in particular if we can find a solution of (3) and (3′) for all $s \in [0, 1]$, with $E(1) > 0$ we will have solved (2).

Remark. If $A \in \mathcal{A}$, then by Lemma 2 all zeros of $M'(A, x)$ are simple. It then follows from the implicit function theorem and from $M'(A, x_k) = 0$, $M''(A, x_k) \neq 0$ that x_k is a locally differentiable function of A.

(4) can be written as

$$F(A(s)) \frac{dB}{ds} = D \tag{6}$$

where

$$B(s) = (A(s), E(s))^T \text{ is a } N+1 \text{ column vector}$$
$$D = (-d_0, \ldots, -d_N)^T \text{ is a } N+1 \text{ column vector.}$$

$F(A(s))$ is a $(N+1) \times (N+1)$ matrix whose k-th row is:

$$\Bigg[K(x_k(A(s)), 0), \ldots, K^{m_t-1}(x_k(A(s)), \xi_t(s)), \sum_{j=0}^{m_1-1} a_{1j}(s)\, K^{j+1}(x_k(A(s)), \xi_1(s)),$$
$$\sum_{j=0}^{m_t-1} a_{tj}(s)\, K^{j+1}(x_k(A(s), \xi_t(s)), -l_k\Bigg].$$

Lemma 4.

$$\det F(A(s)) = \pm K(x, 0) \prod_{i=1}^{t} a_{i,m_i-1}(s) \int_{x_0(A(s))}^{x_1(A(s))} \cdots \int_{x_{N-1}(A(s))}^{x_N(A(s))}$$
$$\times \begin{vmatrix} K_{y_1}(y_1, \bar{\xi}(s))_{N-1} & l_0 - l_1 \\ \vdots & \vdots \\ K_{y_N}(y_N, \bar{\xi}(s))_{N-1} & l_{N-1} - l_N \end{vmatrix} dy_1 \cdots dy_N \tag{8}$$

Proof. By subtracting columns it is clear that

$$\det F(A(s)) = \pm \prod_{i=1}^{t} a_{i,m_i-1}(s) \det P(s)$$

where $P(s)$ is the $(N+1) \times (N+1)$ matrix whose k-th row is:

$$[K(x_k(A(s)), 0), K(x_k(A(s)), \bar{\xi}(s))_{N-1}, -l_k].$$

Recall that $K(x, 0)$ is a constant. We leave the first row of $P(s)$ alone. If we subtract the k-1'th row from the k'th row and apply the fundamental theorem of calculus we get a matrix $C(s)$, whose k'th row for $k > 1$ is:

$$\left[0, \int_{x_{k-1}}^{x_k(A(s))} K_{y_k}(y_k, \bar{\xi}(s))_{N-1}\, dy_k, l_{k-1} - l_k\right].$$

Expanding $C(s)$ by the first column we obtain (8). ∎

Letting $A_0 \in \Lambda$, one sees from (8), using the fact that $\operatorname{sgn}(l_k - l_{k-1}) = (-1)^{N+k}$ and that $K_x(x, \xi)$ is an ECT system, that $\det F(A_0) \neq 0$. Hence near $s = 0$, with initial conditions $A(0) = A$, $E(0) = 0$, (6) can be written

$$\frac{dB(s)}{ds} = F^{-1}(A(s))\, D \tag{9}$$

where the right hand side is a continuously differentiable function of A. Thus for s near 0 (9) has a unique solution. Let $[0, \eta)$ be the maximal interval for which a solution of (9) exists, with $A(s) \in \Lambda$. We wish to show $\eta > 1$. This follows from the following lemma.

Lemma 5. *If $A(s)$, $B(s)$ is a solution of (9) with $A(s) \in \Lambda$ for $s \in [0, \beta)$ $\beta \leqslant 1$, then for some sequence $s_\nu \to \beta$, $A(s_\nu) \to \bar{A}$, $E(s_\nu) \to \bar{E}$, $\bar{A} \in \Lambda$. Hence the solution may be continued beyond β.*

Proof. If the zeros of $M'(A(s_\nu), x)$ are $x_k(s_\nu)$, we pick a subsequence for which all the zeros converge, say $\lim_{s_\nu \to \beta} x_k(s_\nu) = \bar{x}_k$. To show $A(s)$, $E(s)$ is bounded it is sufficient by Lemma 3 to show $a_0(s)$, $E(s)$ are bounded. Assume for example $E(s_\nu) \to \infty$ with $a_0(s_\nu)/E(s_\nu) \to \gamma$, γ possibly infinite. Dividing equation (5) by $E(s_\nu)$ and going to the limit one finds $\gamma K(\bar{x}_k, 0) = l_k$ $k = 0, \dots, N$. Since $K(x, 0)$ is a non-zero constant, this is a contradiction. Thus $E(s_\nu)$ is bounded and again by equation (5), $a_0(s_\nu)$ is bounded.

Hence for some subsequence which we do not relabel all the components of $A(s_\nu)$ and $E(s_\nu)$ converge. Call the limit $\bar{A}$, $\bar{E}$.

By (5) $M(\bar{A}, \bar{x}_k) = \bar{E} l_k + (1 - \beta) d_k$. We will show in Lemma 6 that $\bar{E} > 0$. From this it follows that $M'(\bar{A}, x)$ has $N - 1$ distinct zeros. Thus $\bar{A}$ belongs to Λ, and the solution to the differential equation may be continued beyond β. (For the general proof that the solution may be continued see Hartman [6, Chapt. II, Theorem 3.1 and Lemma 3.1].) ∎

Lemma 6. *If $A(s) \in \Lambda$*

$$\frac{dE(s)}{ds} > 0.$$

Proof. Solving (4) or (6) by Cramer's rule, and applying the method of Lemma 4 to each of the resulting determinants, we find

$$\frac{dE}{ds} = \frac{\displaystyle\int_{x_0(A(s))}^{x_1(A(s))} \cdots \int_{x_{N-1}(A(s))}^{x_N(A(s))} \begin{vmatrix} K_{y_1}(y_1, \xi(s))_{N-1} & d_0 - d_1 \\ \vdots & \vdots \\ K_{y_N}(y_N, \xi(s))_{N-1} & d_{N-1} - d_N \end{vmatrix} dy_1, \dots, dy_N}{\displaystyle\int_{x_0(A(s))}^{x_1(A(s))} \cdots \int_{x_{N-1}(A(s))}^{x_N(A(s))} \begin{vmatrix} K_{y_1}(y_1, \xi(s))_{N-1} & l_0 - l_1 \\ \vdots & \vdots \\ K_{y_N}(y_N, \xi(s))_{N-1} & l_{N-1} - l_N \end{vmatrix} dy_1, \dots, dy_N}.$$

Since $K_x(x, s)$ is a *ECT* system, and $\operatorname{sgn}(d_k - d_{k-1}) = \operatorname{sgn}(l_k - l_{k-1}) = (-1)^{k+N}$ if we expand by the last column we find the numerator and denominator are both of the same sign. ▮

Combining Lemmas 5 and 6, we obtain:

THEOREM 3. *Problem A has a solution.*

We now investigate the uniqueness of the solution.

LEMMA 7. *There exists a continuous map θ of Λ onto V, where V* is the set of (A, E) *that satisfies* (2). (*Note* $\{l_i\}_{i=0}^N$ *is a fixed set.*)

Proof. θ is the mapping obtained from the solution of the differential equation (4), that maps $A(0)$ into $A(1)$, $E(1)$. Since a solution of the differential equation (4) depends continuously on the initial values and parameters the mapping is continuous. If $d_k = l_k$ $k = 0,..., N$, then $A(s) = A(0)$, $E(s) = s$ is the unique solution of (4), showing the mapping is onto.

LEMMA 8. *Each point of V is an isolated point.*

Proof. Consider the system of equations

$$M(A, x_k(A)) - l_k E = 0 \qquad k = 0,..., N,$$

where $x_0 = 0$, $x_N = 1$ and the $x_k(A)$ $k = 1,..., N - 1$ are the zeros of $M'(A, x)$.

The Jacobian $\partial(M(A, x_k) - l_k E)/\partial(A, E)$ is exactly $F(A)$ of (7). By Lemma 4, $\det F(A)$ does not vanish. Hence the implicit function theorem applies to show that locally there is exactly one solution (A, E) of (2). ▮

LEMMA 9. *Given the points $0 < x_1 < x_2 < \cdots < x_{N-1} < 1$ there exists a unique $A \in \Lambda$ (with a_0 specified) so that $M'(A, x_i) = 0$, $i = 1,..., N - 1$.*

Proof. The present situation does not quite satisfy the hypothesis of Karlin and Pincus [7, Theorem 3]. However since $K_x(x, \xi)$ is an *ECT* system the analysis of Karlin and Pincus could be applied in the present situation.

Rather than do this, we present a proof based on the recent paper of D. Barrow [3].

Let $u_i(\xi) = (\partial K/\partial x)(x, \xi)|_{x=x_i}$ $(i = 1,..., N - 1)$ and U be the span of the $\{u_i\}_{i=1}^{N-1}$. Set $\hat{U} = \{u \in U\colon (d^i/d\xi^i)\, u(0) = 0,\ i = 1,..., n - 1\}$. Since $K_x(x, \xi)$ is an *ECT* system it follows that no non-zero member of $\hat{U}$ can have $N - n$ zeroes counting multiplicity in (a, b); that is, $\hat{U}$ is an extended Haar subspace of dimension $N - n$ in (a, b). Let $\{v_i\}_{i=1}^{N-n}$ be a basis for $\hat{U}$. By the results of

[3] there exist unique ξ_i with $a < \xi_1 < \xi_2 < \cdots < \xi_t < b$ and unique $\tilde{a} = (a_{10}, \ldots, a_{1,m_1-1}, a_{20}, \ldots, a_{t,m_t-1})$ such that

$$0 = \int_a^b v_l(\xi)\, d\xi + \sum_{i=1}^{t} \sum_{j=0}^{m_i-1} a_{ij} \frac{d^j}{d\xi^j} v_l(\xi_i) \qquad (l = 1, \ldots, N-n)$$

Further using the fact again that $K_x(x, \xi)$ is an ECT system we can find $n-1$ independent functions $(w_1(\xi), \ldots, w_{n-1}(\xi)) \subset U$ so that

$$\frac{d^j}{d\xi^j} w_l(\xi_i) = 0 \qquad j = 0, \ldots, m_i - 1 \qquad i = 1, \ldots, t \qquad l = 1, \ldots, n-1$$

$$\frac{d^j}{d\xi^j} w_l(0) = \delta_{lj} \qquad \begin{array}{l} j = 1, \ldots, n-1 \\ l = 1, \ldots, n-1 \end{array}.$$

Hence we can find a unique $\hat{a} = (a_1, \ldots, a_{n-1})$ such that

$$0 = \int_a^b w_l(\xi)\, d\xi + \sum_{j=1}^{n-1} a_j w_l^{(j)}(0) \qquad (l = 1, \ldots, n-1). \tag{10'}$$

Further $\{w_1, \ldots, w_{n-1}, v_1, \ldots, v_{N-n}\}$ are independent and span U. Hence corresponding to the set $\{a_0, \tilde{a}, \hat{a}, \bar{\xi}]$ there is a monospline $M(x)$ so that

$$M'(x_i) = 0 \qquad i = 1, \ldots, N-1. \tag{11}$$

If another set $\{a_0, \hat{b}, \tilde{b}, \bar{\zeta}\}$ yields a monospline satisfying (11), then by the uniqueness of (10) and (10'), $\hat{b} = \hat{a}$, $\bar{\zeta} = \bar{\xi}$, $\tilde{b} = \tilde{a}$. ∎

LEMMA 10. *The set A is connected.*

Proof. By lemma 9 elements in A are uniquely characterized by their zeros. Given two monosplines $M(A_1, x)$ and $M(A_2, x)$ with $A_1, A_2 \in$ O, and with $M'(A_i, x)$ having zeros $x_k(A_i)$, we follow Cavaretta [4], in constructing $A(s)$ for $s \in [0, 1]$ such that $M'(A(s), x)$ has zeros:

$$x_k(A(s)) = x_k(A_1) + s(x_k(A_2) - x_k(A_1)) \qquad k = 1, \ldots, N-1. \tag{12}$$

This will establish Lemma 10, for it will show that A is pathwise connected.

To find $A(s)$, we are lead to the system of differential equations:

$$0 = \frac{d}{ds} (M'(A(s), x_k(A_1) + s(x_k(A_2) - x_k(A_1))) \qquad k = 1, \ldots, N-1$$
$$A(0) = A_1 . \tag{13}$$

Written out, this becomes

$$\sum_{j=1}^{n-1} \frac{da_j(s)}{ds} K_x{}^j(x_k(A(s)), 0) + \sum_{i=1}^{t} \sum_{j=0}^{m_i-1} \frac{d}{ds} a_{ij}(s) K_x{}^j(x_k(A(s), \xi_i(s))$$

$$+ \sum_{i=1}^{t} \sum_{j=0}^{m_i-1} a_{ij}(s) K_x^{j+1}(x_k(A(s), \xi_i(s)) \frac{d\xi_i}{ds}$$

$$= -(x_k(A_2) - x_k(A_1)) \Bigg[\sum_{j=1}^{n-1} a_j(s) K_{xx}^j(x_k(A(s), 0)$$

$$+ \sum_{i=1}^{t} \sum_{j=0}^{m_i-1} a_{ij}(s) K_{xx}^j(x_k(A(s), \xi_i(s)) \Bigg] \qquad \begin{array}{l} k = 1, \ldots, N-1 \\ A(0) = A_1 . \end{array}$$

In (14)

$$K_{xx}^j(x, \xi_i) = \frac{\partial^{j+2}}{\partial x^2 \, \partial \xi^j} K(x, \xi)|_{\xi=\xi_i} .$$

We may write this as

$$G(A(s)) \frac{dA}{ds} = H(s) \qquad \text{(with } a_0(s) \text{ not appearing).} \tag{15}$$

If $A \in \Lambda$, one sees from (15) using the fact that $K_x(x, \xi)$ is an *ECT* system that $\det G(A) \neq 0$. Hence near $s = 0$, with initial condition $A(0) = A_1$, (15) can be written

$$\frac{dA}{ds} = G^{-1}(A(s)) H(s) \tag{16}$$

where the right hand side is a continuously differentiable function of A. Thus for s near 0 (16) has a unique solution.

When $A(s)$ exists for $0 = s < \beta = 1$, it follows from (12) that $M'(A(s), x)$ has $N - 1$ distinct zeros in (0, 1). Using Lemmas 2 and 3 and arguing as in Lemma 5 it follows that for some sequence $s_\nu \to \beta$, $\lim A(s_\nu) = \bar{A}$, with $\bar{A} \in \Lambda$. We can infer then that (16) has a solution for all s in some open interval containing [0, 1]. ∎

THEOREM 4. *Problem A has one and only one solution.*

Proof. The result follows if we can show that the set V defined in Lemma 7 consists of one point. V is connected and non-empty, since it is the continuous image of the connected non-empty set Λ by Lemma 7. Lemma 8 asserts that V consists of isolated points. ∎

APPLICATIONS

S. Karlin [9, p. 56] has shown that for each $\alpha > 0$, $K_\alpha(x, \xi) = 1/(1 - x\xi)^\alpha$ is *ETP* for $-1 < x, \xi < 1$; hence, this kernel fits our criterion. Thus

THEOREM 5. *Among all monosplines of the form*

$$M(A, x) = \int_a^b K_\alpha(x, \xi)\, d\xi + \sum_{j=0}^{n-1} K_\alpha{}^j(x, 0) + \sum_{i=1}^{t} \sum_{j=0}^{m_i-1} a_{ij} K_\alpha{}^j(x, \xi_i)$$

with $0 < a < b < d' < 1$, $0 \leqslant \xi_1 < \cdots \leqslant \xi_t \leqslant d'$, m_i *odd, and* $n \geqslant 1$, *there exists one and only one element of minimal uniform norm over* $[0, 1]$. *This minimal monospline is uniquely characterized by a set of* $N + 1$ *points* $0 = x_0 < x_1 < \cdots < x_N = 1$ *such that*

$$M(x_i) = (-1)^{i+N} \| M \| \qquad i = 0,..., N.$$

A similar result holds for the kernel $\exp(x\xi)$.

Finally we remark that our results are still valid if in Assumption I, we have $c = 0$, $d = 1$. However in this case some of the proofs would be more complicated.

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